Quasitriangularity and Enveloping Algebras for Inhomogeneous Quantum Groups

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Abstract

Coquasitriangular universal $\mathcal{R}$ matrices on quantum Lorentz and quantum Poincaré groups are classified. The results extend (under certain assumptions) to inhomogeneous quantum groups of [10]. Enveloping algebras on those objects are described.

0 Introduction

Possible R-matrices for the fundamental representations of inhomogeneous quantum groups were found in Proposition 3.14 of [10]. In the present paper we describe universal $\mathcal{R}$ matrices for those objects (under certain assumptions which are fulfilled in the case of quantum Poincaré groups [11]). Our study will be useful in developing a simple physical model [8] of free particles on a quantum Minkowski space [11].

In Section 1 we show how to construct co(quasi)triangular (*-)bialgebras and Hopf (*-)algebras whose relations are given by means of intertwiners: we simplify and extend results of [7]. Next, in Section 2 we classify co(quasi)triangular (*-)structures $\mathcal{R}$ on quantum Lorentz groups [18]. Using the results of Sections 1 and 2, in Section 3 we classify such structures on quantum Poincaré groups [11] and also, more generally, on inhomogeneous quantum groups [10] (certain natural assumptions regarding mainly restriction of those structures to the homogeneous quantum group are made). In Section 4 we use $\mathcal{R}$ to define enveloping algebras for inhomogeneous quantum groups.

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We sum over repeated indices (Einstein's convention). We work over the field $\mathbf{C}$. We write $a \sim b$ if $a = \lambda b$ for $\lambda \in \mathbf{C}\backslash\{0\}$. If V, W are vector spaces then $\tau : V \otimes W \to W \otimes V$ is given by $\tau(x \otimes y) = y \otimes x$, $x \in V$, $y \in W$. If $\mathcal{A}$ is an algebra, $v \in M_N(\mathcal{A})$, $w \in M_K(\mathcal{A})$, then the tensor product $v \otimes w \in M_{NK}(\mathcal{A})$ is defined by $(v \otimes w)_{ij,kl} = v_{ik}w_{jl}$, $i, k = 1, \ldots, N$, $j, l = 1, \ldots, K$. We set $\dim v = N$. If $\mathcal{A}$ is a *-algebra then the conjugate of v is defined as $\bar{v} \in M_N(\mathcal{A})$ where $\bar{v}_{ij} = v_{ij}{}^*$, $i, j = 1, \ldots, N$.

Throughout the paper quantum groups H are abstract objects described by the corresponding (*-)bialgebras $\mathrm{Poly}(H) = (\mathcal{A}, \Delta)$. We denote by Δ, ε, S the comultiplication, counit and (if exists) coinverse of $\mathrm{Poly}(H)$. We say that v is a representation of H (i.e. $v \in$ Rep H) if $v \in M_N(\mathcal{A})$, $N \in \mathbf{N}$, and $\Delta v_{ij} = v_{ik} \otimes v_{kj}$, $\varepsilon(v_{ij}) = \delta_{ij}$, $i, j = 1, \ldots, N$. The conjugate of a representation and tensor product of representations are also representations. Matrix elements of representations of H span $\mathcal{A}$ as a linear space. The set of nonequivalent irreducible representations of H is denoted by Irr H. If $v, w \in$ Rep H, then we say that $A \in M_{\dim w \times \dim v}(\mathbf{C})$ intertwines v with w (i.e. $A \in$ Mor(v, w)) if $Av = wA$. The dual vector space $\mathcal{A}'$ is an algebra w.r.t. the convolution defined by $\rho * \rho' = (\rho \otimes \rho')\Delta$. It has a unit $I_{\mathcal{A}'} = \varepsilon$. For $\rho \in \mathcal{A}'$, $a \in \mathcal{A}$, we set $\rho * a = (id \otimes \rho)\Delta a$, $a * \rho = (\rho \otimes id)\Delta a$.

1 Coquasitriangular bialgebras

In this section we discuss bialgebras $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ defined by several fundamental representations of G and intertwiners among them. We provide necessary and sufficient conditions for the existence of coquasitriangular structure $\mathcal{R}$ for G. Hopf and $*$ structures are also investigated. The results generalize results of Theorem 1.4 of [7] (see also Theorem 1.1 of [18]).

Proposition 1.1 *Let $\mathcal{B}$ be an algebra generated by w^α_{mn}, $m, n = 1, \ldots, d_\alpha$, $\alpha \in \mathcal{J}$ and relations ($0 \in \mathcal{J}$, $d_0 = 1$)*

$$w^\alpha \otimes w^0 = w^0 \otimes w^\alpha = w^\alpha, \ \alpha \in \mathcal{J}, \tag{1.1}$$

$$W_m(w^{\alpha_{m1}} \otimes w^{\alpha_{m2}} \otimes \ldots w^{\alpha_{ms_m}}) = (w^{\beta_{m1}} \otimes \ldots \otimes w^{\beta_{mt_m}})W_m, \ m \in K. \tag{1.2}$$

Then $w^0 = (I_B)$ and there exists a unique Δ such that $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ *is a bialgebra and w^α, $\alpha \in \mathcal{J}$, representations of G.*

Proof is the same as in Theorem 1.4 of [7]. □

Let us recall

Definition 1.1 (cf. [1], [2], [4]) We say that $(\mathcal{B}, \Delta, \mathcal{R})$ is a coquasitriangular (CQT) bialgebra if $(\mathcal{B}, \Delta)$ is a bialgebra and $\mathcal{R} \in (\mathcal{B} \otimes \mathcal{B})'$ satisfies

$$\mathcal{R}(I \otimes x) = \mathcal{R}(x \otimes I) = \varepsilon(x), \tag{1.3}$$

$$\mathcal{R}(xy \otimes z) = \mathcal{R}(x \otimes z^{(1)})\mathcal{R}(y \otimes z^{(2)}), \tag{1.4}$$

$$\mathcal{R}(x \otimes yz) = \mathcal{R}(x^{(1)} \otimes z)\mathcal{R}(x^{(2)} \otimes y), \tag{1.5}$$

$$y^{(1)}x^{(1)}\mathcal{R}(x^{(2)} \otimes y^{(2)}) = \mathcal{R}(x^{(1)} \otimes y^{(1)})x^{(2)}y^{(2)} \tag{1.6}$$

where we have used the Sweedler's notation $\Delta(x) = x_i^{(1)} \otimes x_i^{(2)} \equiv x^{(1)} \otimes x^{(2)}$.

Remark 1.1 (cf. [5], [4], Proposition 1.3 of [7]) Let $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ be a bialgebra and $\mathcal{R} \in (\mathcal{B} \otimes \mathcal{B})'$. For each $v, w \in \mathrm{Rep}\, G$ one can define $R^{vw} \in \mathrm{Lin}(\mathbf{C}^{\dim v} \otimes \mathbf{C}^{\dim w}, \mathbf{C}^{\dim w} \otimes \mathbf{C}^{\dim v})$ by

$$(R^{vw})_{ij,kl} = \mathcal{R}(v_{jk} \otimes w_{il}),\ \ j,k = 1,\dots,\dim v,\ \ i,l = 1,\dots,\dim w. \tag{1.7}$$

Then

$$(\mathbb{1} \otimes S)R^{v_1 w} = R^{v_2 w}(S \otimes \mathbb{1}) \text{ if } S \in \mathrm{Mor}(v_1, v_2), \tag{1.8}$$

$$(S \otimes \mathbb{1})R^{v w_1} = R^{v w_2}(\mathbb{1} \otimes S) \text{ if } S \in \mathrm{Mor}(w_1, w_2), \tag{1.9}$$

$v, w, v_1, w_1 \in \mathrm{Rep}\, G$. Suppose that $L \subset \mathrm{Rep}\, G$ is such that the matrix elements of representations $v \in L$ linearly span $\mathcal{B}$ (e.g. $L = \mathrm{Rep}\, G$). Consider the conditions

$$R^{0v} = R^{v0} = \mathbb{1},\ \ v \in L, \tag{1.10}$$

$$R^{v_1 \otimes v_2, w} = (R^{v_1 w} \otimes \mathbb{1})(\mathbb{1} \otimes R^{v_2 w}),\ \ v_1, v_2, w \in L, \tag{1.11}$$

$$R^{v, w_1 \otimes w_2} = (\mathbb{1} \otimes R^{v w_2})(R^{v w_1} \otimes \mathbb{1}),\ \ v, w_1, w_2 \in L, \tag{1.12}$$

$$R^{vw} \in \mathrm{Mor}(v \otimes w, w \otimes v),\ \ v, w \in L. \tag{1.13}$$

Then (1.10) $\Leftrightarrow$ (1.3), (1.11) $\Leftrightarrow$ (1.4), (1.12) $\Leftrightarrow$ (1.5), (1.13) $\Leftrightarrow$ (1.6).

Theorem 1.1 *Let* $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ *be a bialgebra defined in Proposition 1.1 and* $R^{\alpha\beta} \in$ *Lin*$(\mathbf{C}^{d_\alpha} \otimes \mathbf{C}^{d_\beta}, \mathbf{C}^{d_\beta} \otimes \mathbf{C}^{d_\alpha})$, $\alpha, \beta \in \mathcal{J}$.

The following are equivalent

1) there exists $\mathcal{R} \in (\mathcal{B} \otimes \mathcal{B})'$ *such that* $(\mathcal{B}, \Delta, \mathcal{R})$ *is a CQT bialgebra and* $R^{\alpha\beta} = R^{w^\alpha w^\beta}$.

2)

$$R^{0\alpha} = R^{\alpha 0} = \mathbb{1},\ \ \alpha \in \mathcal{J}, \tag{1.14}$$

$$(\mathbb{1} \otimes W_m)R^{\alpha_{m1}\cdot\ldots\cdot\alpha_{ms_m},\gamma} = R^{\beta_{m1}\cdot\ldots\cdot\beta_{mt_m},\gamma}(W_m \otimes \mathbb{1}),\ \ m \in K,\ \ \gamma \in \mathcal{J} \setminus \{0\}, \tag{1.15}$$

$$R^{\gamma,\beta_{m1}\cdot\ldots\cdot\beta_{mt_m}}(\mathbb{1} \otimes W_m) = (W_m \otimes \mathbb{1})R^{\gamma,\alpha_{m1}\cdot\ldots\cdot\alpha_{ms_m}},\ \ m \in K,\ \ \gamma \in \mathcal{J} \setminus \{0\}, \tag{1.16}$$

$$R^{\alpha\beta} \in \ Mor(w^\alpha \otimes w^\beta,\ w^\beta \otimes w^\alpha),\ \ \alpha, \beta \in \mathcal{J} \setminus \{0\}, \tag{1.17}$$

where $R^{\delta_1\cdot\ldots\cdot\delta_{k+1},\gamma} = (R^{\delta_1\cdot\ldots\cdot\delta_k,\gamma} \otimes \mathbb{1})(\mathbb{1} \otimes R^{\delta_{k+1}\gamma})$, $R^{\gamma,\delta_1\cdot\ldots\cdot\delta_{k+1}} = (\mathbb{1} \otimes R^{\gamma\delta_{k+1}})(R^{\gamma,\delta_1\cdot\ldots\cdot\delta_k} \otimes \mathbb{1})$, $\gamma, \delta_1, \dots, \delta_{k+1} \in \mathcal{J}$, $k = 1, 2, \dots$.

Moreover, such $\mathcal{R}$ *is unique.*

Remark. In special cases related statements can be found in [6], [4], [2] and Theorem 1.4 of [7]. Cf. [5].

Proof. Assume condition 1). According to Remark 1.1, (1.10)–(1.13) follow. Thus we get (1.14), (1.17). Using (1.8)–(1.9) for $S = W_m$ (see (1.2)) and (1.11)–(1.12), one obtains (1.15)–(1.16) and the condition 2) is proved.

Moreover, $R^{w^\alpha w^\beta} = R^{\alpha\beta}$ uniquely determine R^{vw}, where $v, w \in L_0$,

$$L_0 = \{\text{tensor products of some number of copies of representations } w^\alpha,\ \alpha \in \mathcal{J}\}.$$

Using (1.7) and the fact that the matrix elements of representations from L_0 linearly span $\mathcal{B}$, the uniqueness of $\mathcal{R}$ follows. It remains to prove

2) $\Rightarrow$ 1): Assume condition 2). Using (1.14), we can replace $\mathcal{J}\setminus\{0\}$ by $\mathcal{J}$ in (1.15)-(1.17). We define the homomorphisms $\mathcal{R}^\beta : \mathcal{B} \to M_{d_\beta}(\mathbf{C})$, $\beta \in \mathcal{J}$, by

$$[\mathcal{R}^\beta(w^\alpha_{ij})]_{kl} = R^{\alpha\beta}_{ki,jl}, \quad \alpha \in \mathcal{J} \tag{1.18}$$

(later on we will have $\mathcal{R}^\beta_{kl} = \mathcal{R}(\cdot \otimes w^\beta_{kl})$). They preserve the relations (1.1)–(1.2) due to (1.14)–(1.15). Setting $\alpha = 0$ in (1.18), we show that $\mathcal{R}^\beta$ are unital. Setting $\beta = 0$ in (1.18), one gets $\mathcal{R}^0_{11} = \varepsilon$ (it is true on the generators w^α_{ij}). Hence $\mathcal{R}^0_{11} * \mathcal{R}^\beta_{kl} = \mathcal{R}^\beta_{kl} * \mathcal{R}^0_{11} = \mathcal{R}^\beta_{kl}$. Let

$$\begin{aligned} X &= \{x \in \mathcal{B} : W_{b_1\dots b_{t_m}, a_1\dots a_{s_m}}(\mathcal{R}^{\alpha_{ms_m}}_{a_{s_m}c_{s_m}} * \dots * \mathcal{R}^{\alpha_{m1}}_{a_1c_1})(x) \\ &= (\mathcal{R}^{\beta_{mt_m}}_{b_{t_m}d_{t_m}} * \dots * \mathcal{R}^{\beta_{1t_1}}_{b_1d_1})(x) W_{d_1\dots d_{t_m}, c_1\dots c_{s_m}},\ m \in K\}. \end{aligned}$$

It is straightforward to show that X is an algebra. According to (1.16), $u^\gamma_{kl} \in X$ ($k, l = 1, \dots, d_\gamma,\ \gamma \in \mathcal{J}$). Thus $X = \mathcal{B}$. Hence there exists a linear antihomomorphism $\theta : \mathcal{B} \to \mathcal{B}'$ such that $\theta(w^\beta_{kl}) = \mathcal{R}^\beta_{kl}$, $k, l = 1, \dots, d_\beta$, $\beta \in \mathcal{J}$ (θ preserves (1.1)–(1.2)). Setting $\mathcal{R}(x \otimes y) = [\theta(y)](x)$, $x, y \in \mathcal{B}$, we obtain an $\mathcal{R} \in (\mathcal{B} \otimes \mathcal{B})'$ which satisfies (1.5), (1.3). Moreover, (1.18) yields $R^{\alpha\beta} = R^{w^\alpha w^\beta}$. Let $Y = \{z \in \mathcal{B} : \forall x, y \in \mathcal{B}\ \mathcal{R}(xy \otimes z) = \mathcal{R}(x \otimes z^{(1)})\mathcal{R}(y \otimes z^{(2)})\}$. Then Y is an algebra (we use (1.5)) and $w^\beta_{kl} \in Y$. Hence, $Y = \mathcal{B}$ and (1.4) follows. Thus (Remark 1.1) we get (1.10)–(1.12) for $L = L_0$. That and (1.13) for $v, w \in \{w^\alpha : \alpha \in \mathcal{J}\}$ (see (1.17)) give (1.13) for $L = L_0$, hence (Remark 1.1) (1.6) and the condition 1) is satisfied.

$\square$

Remark 1.2 The unital antihomomorphism $\theta : \mathcal{B} \to \mathcal{B}'$ introduced in the above proof exists for each CQT bialgebra (cf. [5]) and satisfies $(\theta \otimes \theta)\Delta = \Delta'\theta$ where $\Delta' : \mathcal{B}' \to (\mathcal{B} \otimes \mathcal{B})'$ is defined by $\Delta'(\varphi) = \varphi \circ m$, $\varphi \in \mathcal{B}'$, cf. Appendix of [7].

Let us now pass to the Hopf algebra structure.

Proposition 1.2 (cf. [16], Proof of Theorem 1.4.1 **of [7])** *Let* $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ *be a bialgebra and* w, w', w'' *be representations of* G*. Suppose there exist* $E \in \mathrm{Mor}(I, w \otimes w')$, $E' \in$

$\mathrm{Mor}(w''\otimes w, I)$ *such that E is left nondegenerate, i.e.* $E_{i-} = (E_{ij})_{j=1}^{\dim w'}$, $i = 1, \ldots, \dim w$, *are linearly independent and E' is right nondegenerate, i.e.* $E'_{-k} = (E'_{mk})_{m=1}^{\dim w''}$, $k = 1, \ldots, \dim w$, *are linearly independent. Then w^{-1} exists.*

Proposition 1.3 (cf. [16], Proof of Theorem 1.4.1 **of [7])** *Let $(\mathcal{B}, \Delta)$ be the bialgebra defined in Proposition 1.1. Suppose $(u^\alpha)^{-1}$ exist, $\alpha \in \mathcal{J}$. Then $(\mathcal{B}, \Delta)$ is a Hopf algebra.*

A CQT bialgebra $(\mathcal{B}, \Delta, \mathcal{R})$ such that (B, Δ) is a Hopf algebra is called CQT Hopf algebra.

Proposition 1.4 (cf. Proposition 1.3.1.b **of [7])** *Let* $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ *and $(\mathcal{B}, \Delta, \mathcal{R})$ be a CQT Hopf algebra. Then $(R^{vw})^{-1}$ exist for any $v, w \in$* Rep G. *Moreover $\mathcal{R}' = \mathcal{R}(S \otimes id)$ satisfies*

$$\mathcal{R}'(x^{(1)} \otimes y^{(1)})\mathcal{R}(x^{(2)} \otimes y^{(2)}) = \mathcal{R}(x^{(1)} \otimes y^{(1)})\mathcal{R}'(x^{(2)} \otimes y^{(2)}) = (\varepsilon \otimes \varepsilon)(x \otimes y)$$

(i.e. $\mathcal{R}' = \mathcal{R}^{-1}$) and

$$\mathcal{R}'(v_{il} \otimes w_{jk}) = (R^{vw})^{-1}_{ij,kl}, \ i, l = 1, \ldots, \dim v, \ j, k = 1, \ldots, \dim w.$$

We say that $(\mathcal{B}, \Delta)$ is a *-bialgebra if $(\mathcal{B}, \Delta)$ is a bialgebra, $\mathcal{B}$ is a *-algebra and

$$(* \otimes *)\Delta(x) = \Delta(x^*) \tag{1.19}$$

for $x \in \mathcal{B}$. A Hopf algebra which is a *-bialgebra is called Hopf *-algebra.

Proposition 1.5 *Let $(\mathcal{B}, \Delta)$ be the bialgebra defined in Proposition 1.1. Suppose there exists an involution $\sim : \mathcal{J} \to \mathcal{J}$ such that $\tilde{0} = 0$, $d_{\tilde{\alpha}} = d_\alpha$ and $\tilde{W}_m \in$* $\mathrm{Mor}(w^{\tilde{\alpha}_{msm}} \otimes \ldots \otimes w^{\tilde{\alpha}_{m1}}, w^{\tilde{\beta}_{mtm}} \otimes \ldots \otimes w^{\tilde{\beta}_{m1}})$ *where $(\tilde{W}_m)_{i_{t_m} \ldots i_1, j_{s_m} \ldots j_1} = \overline{(W_m)_{i_1 \ldots i_{t_m}, j_1 \ldots j_{s_m}}}$. Then there exists a unique $* : \mathcal{B} \to \mathcal{B}$ such that $(\mathcal{B}, \Delta)$ is a *-bialgebra and $\overline{w^\alpha} = w^{\tilde{\alpha}}$, $\alpha \in \mathcal{J}$ ($-$ was defined in the Introduction).*

Proof. Uniqueness is trivial. Our assumptions imply that $z^\alpha_{ij} = w^{\tilde{\alpha}}_{ij}$ satisfy (1.1)–(1.2) in the conjugate algebra $\mathcal{B}^j$ (conjugate vector space and opposite multiplication). Therefore the desired $*$ exists (we check $*^2 = id$ and (1.19) on the generators w^α_{ij}). □

Proposition 1.6 (cf. the proofs of Proposition 1.3.$d.ii$ **and Theorem** 1.4.6 **of [7])** *Let $(\mathcal{B}, \Delta, \mathcal{R})$ be a CQT bialgebra and* $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ *be a *-bialgebra. Suppose $M \subset$* Rep G *is such that the matrix elements of representations from M generate $\mathcal{B}$ as unital algebra. The following are equivalent:*

1) $\overline{\mathcal{R}(y^* \otimes x^*)} = \mathcal{R}(x \otimes y)$, $x, y \in \mathcal{B}$
2)
$$\overline{(R^{\bar{w}\bar{v}})_{ji,lk}} = R^{vw}_{ij,kl}, \ j, k = 1, \ldots, \dim v, \ i, l = 1, \ldots, \dim w, \ v, w \in M. \tag{1.20}$$

If one of the above conditions is satisfied, $(\mathcal{B}, \Delta, \mathcal{R})$ is called CQT *-bialgebra. If moreover $(\mathcal{B}, \Delta)$ is a Hopf algebra, $(\mathcal{B}, \Delta, \mathcal{R})$ is called CQT Hopf *-algebra (see Definition 1.1 *of [7]).*

Proposition 1.7 (cf. [5]) *Let* $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ *be a bialgebra and* $\mathcal{R} \in (\mathcal{B} \otimes \mathcal{B})'$. *Suppose* $M \subset \ \mathrm{Rep}\, G$ *is such that the matrix elements of representations from* M *generate* $\mathcal{B}$ *as unital algebra. The following are equivalent:*

1) $\mathcal{R}(x^{(1)} \otimes y^{(1)})\mathcal{R}(y^{(2)} \otimes x^{(2)}) = (\varepsilon \otimes \varepsilon)(x \otimes y)$, $x, y \in \mathcal{B}$ *(i.e.* $\mathcal{R}_{21} = \mathcal{R}^{-1}$*)*

2)
$$(R^{vw})^{-1} = R^{wv}, \ v, w \in M. \tag{1.21}$$

If one of the above conditions is satisfied, we replace CQT by CT (cotriangular) in all definitions (cf. [1]).

Proof. Using (1.10)–(1.12), one can assume that (in the condition 2)) matrix elements of representations from M linearly span $\mathcal{B}$. Then in 1) it is enough to consider $x = v_{ij}$, $y = w_{kl}$, $i, j = 1, \ldots, \dim v$, $k, l = 1, \ldots, \dim w$, $v, w \in M$. Then 1) is equivalent to (1.21). □

2 Quasitriangular structures on quantum Lorentz groups

In this section we classify coquasitriangular (*-) structures on quantum Lorentz groups of [18]. For the quantum Lorentz group of [9] examples of such structures were given in [14] and Remark 7 of Section 3 of [7]. We also classify (cf. [17]) coquasitriangular (*-) structures on (complex) $SL_q(2)$ groups and their real forms.

We first recall the definition of Hopf *-algebra corresponding to a quantum Lorentz group [18] essentially repeating the arguments of Theorem 1.1 of [18]. The bialgebra structure of $(\mathcal{A}, \Delta)$ is obtained by the construction of Proposition 1.1 with $\{w^\alpha : \alpha \in \mathcal{J}\} = \{w^0, w, \tilde{w}\}$. Here the relations (1.1) mean that $w^0 = (I_{\mathcal{B}})$ and the relations (1.2) are given by

$$(w \otimes w)E = Ew^0, \tag{2.1}$$

$$(\tilde{w} \otimes \tilde{w})\tilde{E} = \tilde{E}w^0, \tag{2.2}$$

$$E'(w \otimes w) = w^0 E', \tag{2.3}$$

$$\tilde{E}'(\tilde{w} \otimes \tilde{w}) = w^0 \tilde{E}', \tag{2.4}$$

$$X(w \otimes \tilde{w}) = (\tilde{w} \otimes w)X, \tag{2.5}$$

where

$$\tilde{E} = \tau \bar{E}, \ \tilde{E}' = \bar{E}'\tau, \tag{2.6}$$

$E \in \ \mathrm{Lin}(\mathbf{C}, \mathbf{C}^2 \otimes \mathbf{C}^2)$, $E' \in \ \mathrm{Lin}(\mathbf{C}^2 \otimes \mathbf{C}^2, \mathbf{C})$, $X \in \ \mathrm{Lin}(\mathbf{C}^2 \otimes \mathbf{C}^2, \ \mathbf{C}^2 \otimes \mathbf{C}^2)$ satisfy

$$(E' \otimes 1\!\!1)(1\!\!1 \otimes E) = 1\!\!1, \tag{2.7}$$

$$(X \otimes 1\!\!1)(1\!\!1 \otimes X)(E \otimes 1\!\!1) = 1\!\!1 \otimes E, \tag{2.8}$$

$$\tau \bar{X} \tau = \beta^{-1} X, \tag{2.9}$$

$E'E \neq 0$, X is invertible, $\beta \in \mathbf{C}\backslash\{0\}$.

Warning: Our choice of X may differ from the choice of [18] by a multiplicative nonzero constant.

Moreover, (2.7) implies that E and E' are inverse one to another as matrices,

$$(\mathbb{1} \otimes E')(E \otimes \mathbb{1}) = \mathbb{1}. \tag{2.10}$$

Hence, E is left nondegenerate, E' is right nondegenerate, w^{-1} exists (see Proposition 1.2). Similarly, $(\mathbb{1} \otimes \tilde{E}')(\tilde{E} \otimes \mathbb{1}) = \mathbb{1}$, $\tilde{E}$ is left nondegenerate, $\tilde{E}'$ is right nondegenerate, $\tilde{w}^{-1}$ exists. But $(w^0)^{-1} = w^0$ and Proposition 1.3 implies that $(\mathcal{A}, \Delta)$ is a Hopf algebra.

Setting $w^{\sim} = \tilde{w}$, $\tilde{w}^{\sim} = w$, $w^{0\sim} = w^0$, the assumptions of Proposition 1.5 are satisfied and $(\mathcal{A}, \Delta)$ becomes a *-bialgebra where $*$ is defined by $\bar{w} = \tilde{w}$. It has the same Poincaré series as the classical $SL(2, \mathbf{C})$ group (Theorem 1.2 of [18]). We may assume that

1. $E = e_1 \otimes e_2 - q e_2 \otimes e_1$, $q \in \mathbf{C}\backslash\{0, i, -i\}$, $X = \alpha \tau Q$, Q is given by (13)–(19) of [18] ($q = -1$ in (17)–(19)), $\alpha = t^{-1/2}$ for (13), $\alpha = (-t)^{-1/2}$ for (14), $\alpha = q^{1/2}$ for (15), $\alpha = (-q)^{1/2}$ for (16), $\alpha = (s^2 - 1)^{-1/2}$ for (17), $q = (p^2 - 1)^{-1/2}$ for (18), $\alpha = 1/2$ for (19), or

2. $E = e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_1$ (in that case we set $q = 1$), $X = \tau Q$, Q is given by (20)–(21) of [18],

e_1, e_2 is the canonical basis of $\mathbf{C}^2$. Moreover, $\beta = t/|t|$ for (13)–(14), $\beta = q/|q|$ for (15), $\beta = -q/|q|$ for (16), $\beta = -i \operatorname{sgn} \operatorname{Im}(s)$ for (17), $\beta = \operatorname{sgn}(|p| - 1)$ for (18), $\beta = 1$ for (19)–(21). In all cases $\beta^4 = 1$.

We shall find all $\mathcal{R} \in (\mathcal{A} \otimes \mathcal{A})'$ such that $(\mathcal{A}, \Delta, \mathcal{R})$ is a CQT Hopf algebra. So (cf. Theorem 1.1) we need to determine R^{ww}, $R^{\bar{w}\bar{w}}$, $R^{\bar{w}w}$, $R^{w\bar{w}}$ such that 20 relations (1.15)–(1.16) and 4 relations (1.17) are satisfied (we assume (1.14)). We shall use them in the following. Irreducibility of $w \otimes \bar{w}$ (see [18]) and (2.5) give $R^{w\bar{w}} = \varepsilon_X X$, $R^{\bar{w}w} = \varepsilon'_X X^{-1}$, where $\varepsilon_X, \varepsilon'_X \in \mathbf{C}\backslash\{0\}$ (cf. Proposition 1.4). Moreover, $D = R^{ww}$ and $\tilde{D} = \beta R^{\bar{w}\bar{w}}$ must satisfy (2.11) and (2.17)–(2.20) of [11] (with $L, \tilde{L}$ replaced by $D, \tilde{D}$), hence they are given by (2.21)–(2.22) of [11], i.e. $R^{ww} = L_i$, $R^{\bar{w}\bar{w}} = \tau \overline{L_j^{-1}} \tau$, $i, j = 1, 2, 3, 4$, $L_i = q_i(\mathbb{1} + q_i^{-2} E E')$, $q_{1,2} = \pm q^{1/2}$, $q_{3,4} = \pm q^{-1/2}$. Using (2.1), (2.3)–(2.4) of [11], we obtain $\varepsilon_X = \pm 1$, $\varepsilon'_X = \pm 1$, $\beta = \pm 1$. After some calculations one gets that the 24 relations are satisfied. So we get $4 \cdot 4 \cdot 4$ $\mathcal{R}$ for $q \neq \pm 1$, $\beta = \pm 1$, $2 \cdot 2 \cdot 4$ $\mathcal{R}$ for $q = \pm 1$, $\beta = \pm 1$ and no $\mathcal{R}$ for $\beta = \pm i$.

Set $M = \{w, \bar{w}\}$. According to Proposition 1.6, we get a CQT Hopf *-algebra iff 4 relations (1.20) are satisfied iff $\beta = 1$ and $q_j = q_i^{-1}$ ($4 \cdot 4$ $\mathcal{R}$ for $q \neq \pm 1$, $\beta = 1$, $2 \cdot 4$ $\mathcal{R}$ for $q = \pm 1$, $\beta = 1$, no $\mathcal{R}$ for $\beta \neq 1$). According to Proposition 1.7, we get CT Hopf algebra iff 4 relations (1.21) are satisfied iff $q = 1$, $\varepsilon'_X = \varepsilon_X$ ($2 \cdot 2 \cdot 2$ $\mathcal{R}$ for $q = 1$, $\beta = \pm 1$, no $\mathcal{R}$

otherwise). Clearly, we get CT Hopf *-algebra iff $q = \beta = 1$ and $q_j = q_i^{-1}$, $\varepsilon'_X = \varepsilon_X$ ($2 \cdot 2$ $\mathcal{R}$ for $q = \beta = 1$, no $\mathcal{R}$ otherwise).

Let us also recall $SL_q(2)$ groups [15], [1]. The bialgebra structure is obtained by the construction of Proposition 1.1 with $\{w^\alpha : \alpha \in \mathcal{J}\} = \{w^0, w\}$, the relations (1.1) mean that $w^0 = (I_{\mathcal{B}})$ and those of (1.2) are given by (2.1), (2.3), where E, E' are as above with $q \in \mathbf{C}\backslash\{0\}$ (only the case 1.). One gets that $(\mathcal{A}, \Delta)$ is a Hopf algebra.

We shall describe (cf. [17]) all $\mathcal{R} \in (\mathcal{A} \otimes \mathcal{A})'$ such that $(\mathcal{A}, \Delta, \mathcal{R})$ is a CQT Hopf algebra. Due to Theorem 1.1 we should find $D = R^{ww} \in$ Mor$(w \otimes w, w \otimes w)$ satisfying 4 relations (1.15)–(1.16). This means $D = L_i$, $i = 1, 2, 3, 4$, so we get 4 $\mathcal{R}$ for $q \neq \pm 1$, 0, and 2 $\mathcal{R}$ for $q = \pm 1$. In other words $R^{ww} = \pm L_1^{\pm 1}$ where $L_1 = q^{1/2}(\mathbb{1} + q^{-1} E E')$. We get CT Hopf algebras iff $L_1^2 = \mathbb{1}$ (see Proposition 1.7 with $M = \{w\}$) iff $q = 1$.

Let us pass to real forms. Then $(\mathcal{A}, \Delta)$ becomes a Hopf *-algebra where $*$ is defined by $\bar{w} = v^c$, $v = BwB^{-1}$, $B = \mathbb{1}$ for $SU_q(2)$, $B =$ diag$(1, -1)$ for $SU_q(1,1)$, $q \in \mathbf{R}\backslash\{0\}$ (cf. [15], [12], [7]), $\bar{w} = w$ for $SL_q(2, \mathbf{R})$, $|q| = 1$ (cf. [12] and Proposition 1.5). For $SU_q(2)$, $SU_q(1,1)$ we get CQT Hopf *-algebras iff L_1 is hermitian w.r.t. the canonical scalar product in $\mathbf{C}^2$ (see Proposition 1.6.2 with $M = \{w\}$ and the proof of Theorem 1.4.6 of [7]) iff $q > 0$. For $SL_q(2, \mathbf{R})$ we get CQT Hopf *-algebras iff $\tau \bar{L}_1 \tau = L_1$ (see Proposition 1.6.2 with $M = \{w\}$) iff $q = 1$. So we have CT Hopf *-algebras iff $q = 1$ (for each real form).

3 Quasitriangular structures on inhomogeneous quantum groups

For any Hopf algebra Poly$(H) = (\mathcal{A}, \Delta)$ satisfying certain properties one can construct [10] a Hopf algebra Poly$(G) = (\mathcal{B}, \Delta)$ which describes an inhomogeneous quantum group. For certain CQT Hopf algebra structures $(\mathcal{A}, \Delta, \mathcal{R}_{\mathcal{A}})$ we find all CQT Hopf algebra structures $(\mathcal{B}, \Delta, \mathcal{R})$ such that $\mathcal{R}_{|\mathcal{A} \otimes \mathcal{A}} = \mathcal{R}_{\mathcal{A}}$. The *-structures and cotriangularity are studied as well. In particular we find all C(Q)T Hopf (*-) algebra structures on quantum Poincaré groups.

Throughout the Section Poly$(H) = (\mathcal{A}, \Delta)$ is any bialgebra such that

(a) each representation of H is completely reducible,

(b) Λ is an irreducible representation of H,

(c) Mor$(v \otimes w, \Lambda \otimes v \otimes w) = \{0\}$ for any two irreducible representations v, w of H.

Moreover, we assume that f_{ij}, $\eta_i \in \mathcal{A}'$, $i, j = 1, \ldots, N = \dim \Lambda$, are given and satisfy

1. $\mathcal{A} \ni a \to \rho(a) = \begin{pmatrix} f(a) & \eta(a) \\ 0 & \varepsilon(a) \end{pmatrix} \in M_{N+1}(\mathbf{C})$ is a unital homomorphism,

2. $\Lambda_{st}(f_{tr} * a) = (a * f_{st}) \Lambda_{tr}$ for $a \in \mathcal{A}$,

3. $R^2 = \mathbb{1}$ where $R_{ij,sm} = f_{im}(\Lambda_{js})$,

4. $(\Lambda \otimes \Lambda)_{kl,ij}(\tau^{ij} * a) = a * \tau^{kl}$ for $a \in \mathcal{A}$ where $\tau^{ij} = (R-1)_{ij,mn}(\eta_n * \eta_m - \eta_m(\Lambda_{ns})\eta_s + T_{mn}\varepsilon - f_{nb} * f_{ma}T_{ab})$,

5. $A_3\tilde{F} = 0$ where $A_3 = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - R \otimes \mathbb{1} - \mathbb{1} \otimes R + (R \otimes \mathbb{1})(\mathbb{1} \otimes R) + (\mathbb{1} \otimes R)(R \otimes \mathbb{1}) - (R \otimes \mathbb{1})(\mathbb{1} \otimes R)(R \otimes \mathbb{1})$, $\tilde{F}_{ijk,m} = \tau^{ij}(\Lambda_{km})$,

6. $A_3(Z \otimes \mathbb{1} - \mathbb{1} \otimes Z)T = 0$, $RT = -T$ where $Z_{ij,k} = \eta_i(\Lambda_{jk})$.

In particular, 4. and 5. are satisfied if $\tau^{ij} = 0$.

The inhomogeneous quantum group G corresponds to the bialgebra $\mathrm{Poly}(G) = (\mathcal{B}, \Delta)$ defined (cf. Corollary 3.8.a of [10]) as follows: $\mathcal{B}$ is the universal unital algebra generated by $\mathcal{A}$ and y_i, $i = 1, \ldots, N$, with the relations $I_{\mathcal{B}} = I_{\mathcal{A}}$,

$$y_s a = (a * f_{st})y_t + a * \eta_s - \Lambda_{st}(\eta_t * a), \;\; a \in \mathcal{A}, \tag{3.1}$$

$$(R - \mathbb{1})_{kl,ij}(y_i y_j - \eta_i(\Lambda_{js})y_s + T_{ij} - \Lambda_{im}\Lambda_{jn}T_{mn}) = 0. \tag{3.2}$$

Moreover, $(\mathcal{A}, \Delta)$ is a subbialgebra of $(\mathcal{B}, \Delta)$ and $\Delta y_i = \Lambda_{ij} \otimes y_j + y_i \otimes I$ (y_i were denoted by p_i in [10]). In particular, $\mathcal{P} = \begin{pmatrix} \Lambda & y \\ 0 & I \end{pmatrix}$ is a representation of G.

Remark 3.1 If H is a matrix group, Λ its fundamental representation and $(\mathcal{A}, \Delta)$ its corresponding Hopf algebra (generated by Λ_{ij}, Λ^{-1}_{ij}) then, assuming (a)–(c) and setting $f_{ij} = \delta_{ij}\varepsilon$, $\eta_i = 0$, $T = 0$, $(\mathcal{B}, \Delta)$ corresponds to

$$G = H \ltimes \mathbf{R}^N = \left\{ g = \begin{pmatrix} h & a \\ 0 & 1 \end{pmatrix} \in M_{N+1}(\mathbf{C}) : h \in H, \; a \in \mathbf{R}^N \right\},$$

$f(g) = f(h)$, $y_i(g) = a_i$, $f \in \mathcal{A}$, $i = 1, \ldots, N$, $g \in G$.

According to Corollary 3.8.b and the proof of Proposition 3.12 of [10], the bialgebra $(\mathcal{B}, \Delta)$ can be obtained by the construction of Proposition 1.1 with $\{w^\alpha : \alpha \in \mathcal{J}\} = \mathrm{Irr}\; H \cup \{\mathcal{P}\}$. Here the relations (1.1) mean that $w^0 \equiv (I_{\mathcal{A}}) = (I_{\mathcal{B}})$ and the relations (1.2) are given by

$$(\mathcal{P} \otimes \mathcal{P})R_P = R_P(\mathcal{P} \otimes \mathcal{P}), \tag{3.3}$$

$$(\mathcal{P} \otimes w)N_w = N_w(w \otimes \mathcal{P}), \;\; w \in \mathrm{Irr}\; H, \tag{3.4}$$

$$(w \otimes w')S^\alpha_{ww'w''} = S^\alpha_{ww'w''}w'', \;\; w, w', w'' \in \mathrm{Irr}\; H, \;\; \alpha = 1, \ldots, c^{w''}_{ww'}, \tag{3.5}$$

$$\mathcal{P}i = i\Lambda, \tag{3.6}$$

$$s\mathcal{P} = w^0 s, \tag{3.7}$$

where

$$R_P = \begin{pmatrix} R & Z & -R\cdot Z & (R-1\!\!1)T \\ 0 & 0 & 1\!\!1 & 0 \\ 0 & 1\!\!1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ N_w = \begin{pmatrix} G_w, & H_w \\ 0, & 1\!\!1 \end{pmatrix}, \tag{3.8}$$

$(G_w)_{iC,Dj} = f_{ij}(w_{CD})$, $(H_w)_{iC,D} = \eta_i(w_{CD})$, $w \in$ Rep H, $R = G_\Lambda$, $Z = H_\Lambda$, $S^\alpha_{ww'w''}$ ($\alpha = 1,\dots,c^{w''}_{ww'}$) form a basis of Mor$(w'', w\otimes w')$, $i : \mathbf{C}^N \to \mathbf{C}^N \oplus \mathbf{C}$, $s : \mathbf{C}^N \oplus \mathbf{C} \to \mathbf{C}$ are the canonical mappings. In the following we assume that $(\mathcal{A},\Delta)$ is a Hopf algebra. Then (Proposition 3.12 of [10]) $(\mathcal{B},\Delta)$ is also a Hopf algebra and G_w^{-1} exist:

$$(G_w^{-1})_{Ak,lB} = f_{kl}(w^{-1}_{AB}). \tag{3.9}$$

If $(\mathcal{A},\Delta)$ is a Hopf *-algebra then we also assume $\bar\Lambda = \Lambda$,

$$f_{ij}(S(a^*)) = \overline{f_{ij}(a)},\ \eta_i(S(a^*)) = \overline{\eta_i(a)},\ i,j = 1,\dots,N,\ a\in\mathcal{A}, \tag{3.10}$$

$\tilde T - T \in$ Mor$(w^0, \Lambda\otimes\Lambda)$, where $\tilde T_{ij} = \overline{T_{ji}}$. Then [10] $(\mathcal{B},\Delta)$ has a unique Hopf *-algebra structure such that $(\mathcal{A},\Delta)$ is its Hopf *-subalgebra and $y_i^* = y_i$.

In the following we assume Mor $(I, \Lambda\otimes\Lambda) \cap$ ker $(R + 1\!\!1) = \{0\}$, i.e. Mor $(I,\Lambda\otimes\Lambda) \subset$ ker $(R - 1\!\!1)$. Then (using (4.14) of [10]) $\tilde T = T$. The main result of the present paper is contained in

Theorem 3.1 *Let* Poly$(H) = (\mathcal{A},\Delta)$, Poly$(G) = (\mathcal{B},\Delta)$ *be as above,* $(\mathcal{A},\Delta,\mathcal{R}_\mathcal{A})$ *be a CQT Hopf algebra such that*

$$R^{v\Lambda} = c_v G_v,\ R^{\Lambda v} = c'_v G_v^{-1},\ v\in \text{ Irr } H, \tag{3.11}$$

$c_v, c'_v \in \mathbf{C}\backslash\{0\}$. *We are interested in CQT Hopf algebra structures* $(\mathcal{B},\Delta,\mathcal{R})$ *such that*

$$\mathcal{R}_{|\mathcal{A}\otimes\mathcal{A}} = \mathcal{R}_\mathcal{A}. \tag{3.12}$$

One has:

1. *Such a structure exists iff*
$$\tau^{ij} = 0,\ i,j = 1,\dots,N, \tag{3.13}$$
$$R^{v\Lambda} = G_v,\ R^{\Lambda v} = G_v^{-1},\ v\in \text{ Irr } H. \tag{3.14}$$

2. *Suppose (3.13)–(3.14). Then such structures are in one to one correspondence with* $m \in$ Mor$(w^0, \Lambda\otimes\Lambda)$ *satisfying*
$$(f_{jb} * f_{ia})m_{ab} = m_{ij}\varepsilon, \quad i,j = 1,\dots,N, \tag{3.15}$$

and are determined by

$$R^{vw} = R^{vw} \text{ for } \mathcal{R}_{\mathcal{A}},\ v,w \in \text{ Irr } H, \tag{3.16}$$

$$R^{v\mathcal{P}} = N_v,\ R^{\mathcal{P}v} = N_v^{-1},\ v \in \text{ Irr } H, \tag{3.17}$$

$$R^{\mathcal{P}\mathcal{P}} = R_P + m_P \tag{3.18}$$

where

$$m_P = \begin{pmatrix} 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

3. *Let $\mathcal{R}$ be as in* 2. *and let $(\mathcal{A},\Delta)$, $(\mathcal{B},\Delta)$ be Hopf *–algebras as in the text before the Theorem. Then $(\mathcal{B},\Delta,\mathcal{R})$ is a CQT Hopf *-algebra iff $(\mathcal{A},\Delta,\mathcal{R}_{\mathcal{A}})$ is a CQT Hopf *-algebra and*

$$m_{ij} = \overline{m_{ji}}, \quad i,j = 1,\ldots,N. \tag{3.20}$$

4. *Let $\mathcal{R}$ be as in* 2. *Then $(\mathcal{B},\Delta,\mathcal{R})$ is a CT Hopf algebra iff $(\mathcal{A},\Delta,\mathcal{R}_{\mathcal{A}})$ is a CT Hopf algebra and $m = 0$.*

Proof. **Ad 1–2**. Each such structure is (see Theorem 1.1) uniquely determined by R^{vw}, $R^{v\mathcal{P}}$, $R^{\mathcal{P}v}$ and $R^{\mathcal{P}\mathcal{P}}$ satisfying (1.15)–(1.17) (we assume (1.14)), $v,w \in$ Irr H. Using (3.12), we get (3.16). In virtue of the properties of $\mathcal{R}_{\mathcal{A}}$ the formula (1.17) for R^{vw} follows. Moreover, (1.15)–(1.16) for (3.6) and $w^\gamma = v \in$ Irr H mean $R^{v\mathcal{P}}(1\!\!1 \otimes i) = (i \otimes 1\!\!1)R^{v\Lambda}$, $R^{\mathcal{P}v}(i \otimes 1\!\!1) = (1\!\!1 \otimes i)R^{\Lambda v}$. That and (3.11) give

$$R^{v\mathcal{P}} = c_v \begin{pmatrix} G_v, & ? \\ 0, & ? \end{pmatrix} = c_v N_v,\ R^{\mathcal{P}v} = c'_v \begin{pmatrix} G_v^{-1} & ? \\ 0 & ? \end{pmatrix} = c'_v N_v^{-1}$$

where the second equalities follow from (1.17) for $R^{v\mathcal{P}}, R^{\mathcal{P}v}$, (3.4), the independence of $1, y_i$ $(i = 1,\ldots,N)$ over $\mathcal{A}$ (in left and also in right module, see Corollary 3.6 and (1.4) of [10]) and the condition (c). Using (1.15)–(1.16) for (3.7) and $w^\gamma = v \in$ Irr H, one obtains $c_v = c'_v = 1$, we get (3.14), (3.17). In virtue of (1.15)–(1.16) for (3.6)–(3.7) and $w^\gamma = \mathcal{P}$

$$R^{\mathcal{P}\mathcal{P}} = \begin{pmatrix} R & Z & -R\cdot Z & ? \\ 0 & 0 & 1\!\!1 & 0 \\ 0 & 1\!\!1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R_P + m_P$$

for some $m \in$ Mor$(w^0, \Lambda \otimes \Lambda)$ where the second equality uses (1.17) for $R^{\mathcal{P}\mathcal{P}}$, (3.3) and (3.19). Thus (3.18) follows. We set $R_Q = R_P + m_P = R^{\mathcal{P}\mathcal{P}}$ and replace (3.3) by equivalent (assuming (3.5)) relation

$$(\mathcal{P} \otimes \mathcal{P})R_Q = R_Q(\mathcal{P} \otimes \mathcal{P}). \tag{3.21}$$

The relations (1.15)–(1.16) for (3.21), (3.4) and any $w^\gamma \in$ Irr $H \cup \{\mathcal{P}\}$ are equivalent to

$$(R_Q \otimes 1\!\!1)(1\!\!1 \otimes R_Q)(R_Q \otimes 1\!\!1) = (1\!\!1 \otimes R_Q)(R_Q \otimes 1\!\!1)(1\!\!1 \otimes R_Q), \tag{3.22}$$

$$(1\!\!1 \otimes N_v)(N_v \otimes 1\!\!1)(1\!\!1 \otimes R_Q) = (R_Q \otimes 1\!\!1)(1\!\!1 \otimes N_v)(N_v \otimes 1\!\!1),\ v \in \text{ Irr } H, \tag{3.23}$$

$$(1\!\!1 \otimes R^{vw})(N_v \otimes 1\!\!1)(1\!\!1 \otimes N_w) = (N_w \otimes 1\!\!1)(1\!\!1 \otimes N_v)(R^{vw} \otimes 1\!\!1),\ v, w \in \text{ Irr } H. \tag{3.24}$$

According to Proposition 3.14 of [10] and its proof, (3.22) is equivalent to $\tilde{F} = 0$. Let us denote the standard basis elements in $\mathbf{C}^{\dim v}$, $v \in$ Irr H, by h_i^v, $i = 1, \ldots, \dim v$, $e_i = h_i^\Lambda$ and in $\mathbf{C}$ by $f = 1$. Using (3.65) of [10] and

$$\begin{array}{rcl} N_v(h_i^v \otimes e_j) & = & (G_v)_{kl,ij} e_k \otimes h_l^v, \\ N_v(h_i^v \otimes f) & = & f \otimes h_i^v + (H_v)_{kl,i} e_k \otimes h_l^v, \end{array}$$

we find that (3.23) on $h_i^v \otimes e_j \otimes e_k$ follows from the last formula before Proposition 3.14 in [10], on $h_i^v \otimes e_j \otimes f$, $h_i^v \otimes f \otimes e_j$ follows from (2.18) of [10], on $h_i^v \otimes f \otimes f$ is equivalent (using $Rm = m$) to $\tau^{ij}(v_{AB}) = 0$ and (3.15) applied to v_{AB} for all i, j, A, B. So (3.23) is equivalent to (3.13) (which implies $\tilde{F} = 0$) and (3.15). We also get that (3.24) on $h_i^v \otimes h_j^w \otimes e_k$ follows from $(1\!\!1 \otimes R^{vw})(R^{v\Lambda} \otimes 1\!\!1)(1\!\!1 \otimes R^{w\Lambda}) = (R^{w\Lambda} \otimes 1\!\!1)(1\!\!1 \otimes R^{v\Lambda})(R^{vw} \otimes 1\!\!1)$ (it can be obtained using (1.13), (1.8) for $\mathcal{R}_\mathcal{A}$– cf. [1], [4], (1.14) of [7]), on $h_i^v \otimes h_j^w \otimes f$ follows from the equality obtained by acting η_i on (1.17) for R^{vw} and using condition 1. (see the beginning of the Section).

The relations (1.15)–(1.16) for (3.5) and $w^\gamma \in$ Irr H follow from (1.8), (1.11) for $\mathcal{R}_\mathcal{A}$ while for $w^\gamma = \mathcal{P}$ are equivalent to

$$(N_w \otimes 1\!\!1)(1\!\!1 \otimes N_{w'})(S^\alpha_{ww'w''} \otimes 1\!\!1) = (1\!\!1 \otimes S^\alpha_{ww'w''})N_{w''}.$$

That on $h_i^{w''} \otimes e_j$ follows from (1.8), (1.11) for $\mathcal{R}_\mathcal{A}$, on $h_i^{w''} \otimes f$ follows from the equality obtained by acting η_i on (3.5).

Ad 3. We need to check (1.20) for $M =$ Irr $H \cup \{\mathcal{P}\}$. For R^{vw}, $v, w \in$ Irr H, it is equivalent to the fact that $(\mathcal{A}, \Delta, \mathcal{R}_\mathcal{A})$ is a CQT Hopf *-algebra, for $R^{\mathcal{P}v}$ and $R^{v\mathcal{P}}$ follows from (3.9), (3.10) and the properties of η_i, for $R^{\mathcal{P}\mathcal{P}}$ it is equivalent (using (4.14) and the next formula of [10], $Rm = m$, $RT = -T$) to (3.20).

Ad 4. We need to check (1.21) for $M =$ Irr $H \cup \{\mathcal{P}\}$. For R^{vw}, $v, w \in$ Irr H, it is equivalent to the fact that $(\mathcal{A}, \Delta, \mathcal{R}_\mathcal{A})$ is a CT Hopf algebra, for $R^{\mathcal{P}v}$ and $R^{v\mathcal{P}}$ follows from (3.17), for $R^{\mathcal{P}\mathcal{P}}$ it is equivalent (using $R_P^2 = 1\!\!1$) to $m = 0$. □

Remark. If the first formula of the condition 6. is replaced by $0 \neq A_3(Z \otimes 1\!\!1 - 1\!\!1 \otimes Z)T \in$ Mor $(I, \Lambda \otimes \Lambda \otimes \Lambda)$ (this is allowed by [10]) then (3.22) is not satisfied and there is no CQT Hopf algebra structure on $(\mathcal{B}, \Delta)$.

As an application we shall consider $(\mathcal{A},\Delta) =$ Poly(L) where L is a quantum Lorentz group. The corresponding inhomogeneous quantum groups are called quantum Poincaré groups and are (almost) classified in [11]. The classification of C(Q)T Hopf (*-) algebra structures on them is given in

Theorem 3.2 *Let* Poly$(P) = (\mathcal{B},\Delta)$ *be the Hopf* **-algebra corresponding to a quantum Poincaré group* P *[11] described by an admissible choice of quantum Lorentz group (cases 1)–7)),* $s=\pm 1$, H *and* T.

1. *Let us consider CQT Hopf algebra structures* $(\mathcal{B},\Delta,\mathcal{R})$ *on* P. *One has:*

a) *In the cases* 1) *(except* $s=1$, $t=1$, $t_0\neq 0$*— see Remark* 1.8 *of [11]* [1] *),* 2)*,* 3)*,* 4) *(except* $s=1$, $b\neq 0$*), 5) (except* $s=\pm 1$, $t=1$, $t_0\neq 0$*), 6), 7) each such structure is uniquely determined by*

$$R^{ww}=kL,\ R^{w\bar w}=kX,\ R^{\bar w w}=qkX^{-1},\ R^{\bar w\bar w}=qk\tilde L$$

and (3.17)–(3.19), where

$$m=cm_0,\ m_0=(V^{-1}\otimes V^{-1})(\mathbb{1}\otimes X\otimes\mathbb{1})(E\otimes\tau E),\ L=sq^{1/2}(\mathbb{1}+q^{-1}EE'),\ \tilde L=q\tau L\tau,$$

where $s=\pm 1$, E,E' *are fixed for fixed* P *and given in [11],* $k=\pm 1$ *(two possible* $\mathcal{R}$ *for each* $c\in\mathbf{C}$*).*

b) *In the other cases there is no such structure.*

2. *Let* $\mathcal{R}$ *be as in* 1. *We get CQT Hopf* **-algebra iff* $q=1$ *(which excludes the cases 5), 6), 7)) and* $c\in\mathbf{R}$.

3. *Let* $\mathcal{R}$ *be as in* 1. *We get CT Hopf algebra iff* $q=1$ *(which excludes the cases 5), 6), 7)) and* $c=0$ *(then it is also CT Hopf* **-algebra).*

Proof. **Ad 1**. We shall use Theorem 3.1, the results of [11] and Section 2. Thus H is a quantum Lorentz group, $\Lambda=V^{-1}(w\otimes\bar w)V$ with $V_{CD,i}=(\sigma_i)_{CD}$ ($\sigma_0=\mathbb{1},\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices), $q=\beta=\pm 1$, the assumptions about H and G are satisfied. Moreover, $G_w=(V^{-1}\otimes\mathbb{1})(\mathbb{1}\otimes X)(L\otimes\mathbb{1})(\mathbb{1}\otimes V)$, $G_{\bar w}=(V^{-1}\otimes\mathbb{1})(\mathbb{1}\otimes\tilde L)(X^{-1}\otimes\mathbb{1})(\mathbb{1}\otimes V)$, where $L=sq^{1/2}(\mathbb{1}+q^{-1}EE')$ and $\tilde L=q\tau\overline{L^{-1}}\tau=q\tau L\tau$. The possible η and T are described in [11]. According to the results of Section 2, each CQT Hopf algebra structure on $(\mathcal{A},\Delta)$ is uniquely characterized by $R^{ww}=\varepsilon_L L$, $R^{\bar w\bar w}=\varepsilon'_L\tilde L$, $R^{w\bar w}=\varepsilon_X X$, $R^{\bar w w}=\varepsilon'_X X^{-1}$, where $\varepsilon_L^2=\varepsilon_L'^2=\varepsilon_X^2=\varepsilon_X'^2=1$ (16 possible $\mathcal{R}_{\mathcal{A}}$). Using (1.8)–(1.9), (1.11)–(1.12), one gets (3.11) for $v=w,\bar w$ with $c_w=\varepsilon_L\varepsilon_X$, $c_{\bar w}=\varepsilon'_L\varepsilon'_X$, $c'_w=q\varepsilon_L\varepsilon'_X$, $c'_{\bar w}=q\varepsilon_X\varepsilon'_L$. In virtue of Proposition 2.1 of [11] and (1.8)–(1.9), (1.11)–(1.12) the condition (3.11) is satisfied for all $v\in$ Irr H.

[1] In the old version of Remark 1.8 of [11] one should replace t by t_0 (except of expressions $t=1$). That t_0 is identified with t of (3)–(4) of Ref. 16 of [11].

Due to Proposition 3.13.2 and Proposition 4.8 of [10], (3.13) is equivalent to $\tau^{ij}(w_{AB}) = 0$, $i, j = 0, 1, 2, 3$, $A, B = 1, 2$, which means (cf. the proof of Theorem 1.6 of [11]) $\lambda = 0$, which excludes the case 4), $s = 1$, $b \neq 0$, the case 1), $s = 1$, $t = 1$, $t_0 \neq 0$ and the case 5), $s = \pm 1$, $t = 1$, $t_0 \neq 0$ where $t_0 \in \mathbf{R}$ is introduced in Remark 1.8 of [11]. Moreover, (3.14) means that $\varepsilon'_X = \varepsilon'_L = qk$, $\varepsilon_X = \varepsilon_L = k$ for some $k = \pm 1$. Using Theorem 3.1.1–2, $\mathrm{Mor}(w^0, \Lambda \otimes \Lambda) = \mathbf{C} m_0$ and (3.15) for $m = m_0$ (it is enough to prove it on w_{AB}, ${w_{AB}}^*$ when it follows from the 20 relations considered in Section 2), we get 1.

Ad 2. We use $q = \beta = 1$ (which implies $q_j = q_i^{-1}$), $(m_0)_{ij} = \overline{(m_0)_{ji}}$ and Theorem 3.1.3.

Ad 3. We use $q = 1$ (which implies $\varepsilon'_X = \varepsilon_X$) and Theorem 3.1.4. $\square$

4 Enveloping algebras

In this section we study enveloping algebras of inhomogeneous quantum groups. We assume that $(\mathcal{A}, \Delta, \mathcal{R}_{\mathcal{A}})$ and $(\mathcal{B}, \Delta, \mathcal{R})$ are CQT Hopf algebras as in Theorem 3.1.1–2 (e.g. as in Theorem 3.2.1).

We essentially follow the scheme of [12] and [13] but now we don't assume $Z = T = 0$. We define $l_{jl} \in \mathcal{B}'$, $j, l = 1, \ldots, N, +$, by

$$l_{jl}(x) = \mathcal{R}(x \otimes \mathcal{P}_{jl}) \tag{4.1}$$

(in CT case l corresponds to $L^{\pm}$ of [12] on the subalgebra generated by $\mathcal{P}_{ac}$). According to (1.5) and (1.13) for $v = w = \mathcal{P}$,

$$\begin{aligned}
R^{\mathcal{P}\mathcal{P}}_{ab,cd} l_{df}(x^{(1)}) l_{ce}(x^{(2)}) &= R^{\mathcal{P}\mathcal{P}}_{ab,cd} \mathcal{R}(x^{(1)} \otimes \mathcal{P}_{df}) \mathcal{R}(x^{(2)} \otimes \mathcal{P}_{ce}) \\
&= \mathcal{R}(x \otimes R^{\mathcal{P}\mathcal{P}}_{ab,cd} \mathcal{P}_{ce} \mathcal{P}_{df}) \\
&= \mathcal{R}(x \otimes \mathcal{P}_{ac} \mathcal{P}_{bd} R^{\mathcal{P}\mathcal{P}}_{cd,ef}) \\
&= \mathcal{R}(x^{(1)} \otimes \mathcal{P}_{bd}) \mathcal{R}(x^{(2)} \otimes \mathcal{P}_{ac}) R^{\mathcal{P}\mathcal{P}}_{cd,ef} \\
&= l_{bd}(x^{(1)}) l_{ac}(x^{(2)}) R^{\mathcal{P}\mathcal{P}}_{cd,ef},
\end{aligned}$$

hence

$$R^{\mathcal{P}\mathcal{P}}_{ab,cd} (l_{df} * l_{ce}) = (l_{bd} * l_{ac}) R^{\mathcal{P}\mathcal{P}}_{cd,ef}, \ a, b, e, f = 1, \ldots, N, +. \tag{4.2}$$

Setting $l_{ab} = L_{ab}$, $l_{a+} = M_a$, and using $l_{+a} = 0$, $l_{++} = \varepsilon$, $a, b = 1, \ldots, N$, and (3.18), (4.2) is equivalent to

$$R_{ab,cd}(L_{df} * L_{ce}) = (L_{bd} * L_{ac}) R_{cd,ef}, \tag{4.3}$$

$$R_{ab,cd}(M_d * L_{ce}) + Z_{ab,c} L_{ce} = (L_{bd} * L_{ac}) Z_{cd,e} + L_{be} * M_a, \tag{4.4}$$

$$R_{ab,cd}(L_{df} * M_c) - (RZ)_{ab,d} L_{df} = -(L_{bd} * L_{ac})(RZ)_{cd,f} + M_b * L_{af}, \tag{4.5}$$

$$R_{ab,cd} M_d * M_c + Z_{ab,c} M_c - (RZ)_{ab,d} M_d + s_{ab} \varepsilon = (L_{bd} * L_{ac}) s_{cd} + M_b * M_a, \tag{4.6}$$

$a, b, e, f = 1, \ldots, N$, where $s = (R-1)T + m$. Let us notice that (4.5) follows from (4.3)–(4.4). Moreover, using (1.4), (1.3), $l_{ac}(xy) = l_{ab}(x) l_{bc}(y)$, $l_{ac}(I) = \delta_{ac}$, $a, c = 1, \ldots, N, +$, $x, y \in \mathcal{B}$. Thus $L_{ac}(I) = \delta_{ac}$, $M_a(I) = 0$,

$$L_{ac}(xy) = L_{ab}(x) L_{bc}(y), \tag{4.7}$$

$$M_a(xy) = L_{ab}(x) M_b(y) + M_a(x)\varepsilon(y), \tag{4.8}$$

$a, b = 1, \ldots, N$, $x, y \in \mathcal{B}$. Also $l_{jl}(\mathcal{P}_{ab}) = \mathcal{R}(\mathcal{P}_{ab} \otimes \mathcal{P}_{jl}) = R^{\mathcal{P}\mathcal{P}}_{ja,bl}$, $l_{jl}(w_{AB}) = \mathcal{R}(w_{AB} \otimes \mathcal{P}_{jl}) = R^{w\mathcal{P}}_{jA,Bl} = (N_w)_{jA,Bl}$ (see (4.1), (1.7), (3.17), (3.8)), $a, b, j, l = 1, \ldots, N, +$, $A, B = 1, \ldots, \dim w$, $w \in$ Rep H. Therefore

$$L_{jl}(\Lambda_{ab}) = R_{ja,bl}, \tag{4.9}$$

$$L_{jl}(y_a) = -(RZ)_{ja,l}, \tag{4.10}$$

$$L_{jl}(w_{AB}) = (G_w)_{jA,Bl}, \tag{4.11}$$

$$M_j(\Lambda_{ab}) = Z_{ja,b}, \tag{4.12}$$

$$M_j(y_a) = s_{ja}, \tag{4.13}$$

$$M_j(w_{AB}) = (H_w)_{jA,B}, \tag{4.14}$$

$a, b, j, l = 1, \ldots, N$, $A, B = 1, \ldots, \dim w$, $w \in$ Rep H. It is clear that l_{jl} generate a unital subalgebra of $\mathcal{B}'$ (w.r.t. convolution $*$). Endowing it with Δ' of Remark 1.2, we get a bialgebra U with l as its corepresentation. Adding $l^{(m)}_{ij} = l_{ij} \circ S^m$, one obtains a Hopf algebra $\hat{U}$ with coinverse $S'(l^{(m)}) = l^{(m+1)}$ and corepresentations $l^{(2k)}$, $(l^{(2k+1)})^T$, $k = 0, 1, 2, \ldots$. Acting S'^m on (4.2), one obtains

$$\begin{aligned} R^{\mathcal{P}\mathcal{P}}_{ab,cd}(l^{(2k)}_{df} * l^{(2k)}_{ce}) &= (l^{(2k)}_{bd} * l^{(2k)}_{ac}) R^{\mathcal{P}\mathcal{P}}_{cd,ef}, \\ R^{\mathcal{P}\mathcal{P}}_{ab,cd}(l^{(2k+1)}_{ce} * l^{(2k+1)}_{df}) &= (l^{(2k+1)}_{ac} * l^{(2k+1)}_{bd}) R^{\mathcal{P}\mathcal{P}}_{cd,ef}. \end{aligned}$$

$\hat{U}$ is called enveloping algebra of $(\mathcal{B}, \Delta)$. It can be sometimes too small. It happens e.g. in the classical case (see Remark 3.1) with $\mathcal{R} = \varepsilon \otimes \varepsilon$ when $\hat{U} = \mathbf{C}\varepsilon$. Cf. also [3].

Notice that $L_{jl}|_{\mathcal{A}} = f_{jl}$, $M_j|_{\mathcal{A}} = \eta_a$, $l_{|\mathcal{A}} = \rho$. According to the proof of Theorem 1.1, there exists antihomomorphism $\theta : \mathcal{B} \to \mathcal{B}'$ (given by $\mathcal{R}$) such that $\theta(\Lambda_{jl}) = L_{jl}$, $\theta(y_j) = M_j$, $\theta(I) = \varepsilon$. Therefore the formulae (3.60), (3.46) and (1.12) of [10] yield (4.4), (4.6) (with s replaced by $(R-1)T$) and (4.3) which give

$$f_{be} * \eta_a = R_{ab,cd}\eta_d * f_{ce} + Z_{ab,c} f_{ce} - (f_{bd} * f_{ac}) Z_{cd,e}$$

(cf. (2.18) of [10]), the condition $\tau^{ij} = 0$ and the last formula before Proposition 3.14 in [10].

Suppose $(\Lambda \otimes \Lambda)k = kw^0$, $n(\Lambda \otimes \Lambda) = w^0 n$ $(w^0 = (I_{\mathcal{B}}))$. Applying θ, we get

$$(L_{bd} * L_{ac})k_{cd} = k_{ab}\varepsilon,\ n_{ab}(L_{bd} * L_{ac}) = n_{cd}\varepsilon,\ a,b,c,d = 1,\ldots,N. \tag{4.15}$$

Let us set $X_{lj} = L_{jl} \circ S \in \hat{U}$, $j,l = 1,\ldots,N$.
Then

$$X_{ik}(xy) = X_{ij}(x)X_{jk}(y),\ X_{ik}(I) = \delta_{ik},\ x,y \in \mathcal{B}, \tag{4.16}$$

$$X_{ik}(a) = f_{ki}(S(a)),\ a \in \mathcal{A},\ i,k = 1,\ldots,N. \tag{4.17}$$

Using the last equation in the proof of Proposition 3.12 of [10], (4.7), (3.9) and (4.10), we obtain

$$X_{ik}(y_l) = Z_{lk,i}. \tag{4.18}$$

Moreover, (4.15) yields

$$k_{ab}(X_{ac} * X_{bd}) = k_{cd}\varepsilon,\ (X_{ac} * X_{bd})n_{cd} = n_{ab}\varepsilon. \tag{4.19}$$

As in the proof of Proposition 3.1.2 of [8], there exists a unital homomorphism $X : \mathcal{B} \to M_{N+1}(\mathbf{C})$ such that

$$X = \begin{pmatrix} (X_{jl})_{j,l=1}^N & (Y_j)_{j=1}^N \\ 0 & \varepsilon \end{pmatrix}$$

for some $Y_j \in \mathcal{B}'$ satisfying $Y_j(a) = 0$, $a \in \mathcal{A}$, $Y_j(y_k) = \delta_{jk}$, $j,k = 1,\ldots,N$. Setting $X_{j+} = Y_j$, $X_{+j} = 0$, $X_{++} = \varepsilon$, $j = 1,\ldots,N$, the commutation relations among X_{ij}, $i,j = 1,\ldots,N,+$, are the same as in (3.7) of [8], i.e.

$$(X_{ab} * X_{cd})K_{bd,st} = K_{ac,bd}(X_{bs} * X_{dt}),\ a,c,s,t = 1,\ldots,N,+, \tag{4.20}$$

where

$$K = \begin{pmatrix} R^T & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(it is also possible to replace K in (4.20) by $K + n_P$, where $n \in \operatorname{Mor}(\Lambda \otimes \Lambda, w^0)$ – see (3.19), (4.19)).

Defining $X_{ij}^{(m)} = X_{ij} \circ S^m$, $i,j = 1,\ldots,N,+$, $m = 0,1,2,\ldots$, one gets a Hopf algebra $\hat{V}$ generated (as an algebra) by $l_{ij}^{(m)}$ and $X_{ij}^{(m)}$. Clearly $S'(X^{(m)}) = X^{(m+1)}$; $X^{(2k)}$, $[X^{(2k+1)}]^T$, $k = 0,1,\ldots$, are corepresentations of $\hat{V}$. Letting S' act on (4.20), one obtains

$$\begin{aligned} (X_{ab}^{(2k)} * X_{cd}^{(2k)})K_{bd,st} &= K_{ac,bd}(X_{bs}^{(2k)} * X_{dt}^{(2k)}), \\ (X_{cd}^{(2k+1)} * X_{ab}^{(2k+1)})K_{bd,st} &= K_{ac,bd}(X_{dt}^{(2k+1)} * X_{bs}^{(2k+1)}). \end{aligned}$$

$\hat{V}$ is called enlarged enveloping algebra of $(\mathcal{B}, \Delta)$. It would be interesting to find the commutation relations between M_i and Y_j.

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